Alternative Boundary Condition Implementations for Crank Nicolson Solution to the Heat Equation

ME 448/548 Notes

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Overview

1. Goal is to allow Dirichlet, Neumann and mixed boundary conditions
2. Use ghost node formulation
   - Preserve spatial accuracy of $\mathcal{O}(\Delta x^2)$
   - Preserve tridiagonal structure to the coefficient matrix
3. Implement in a code that uses the Crank-Nicolson scheme.
4. Demonstrate the technique on sample problems

Mixed Boundary Condition

The general form of a mixed boundary condition is

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = g_0 + h_0 u \tag{1}$$

This accommodates Neumann boundary conditions ($\partial u/\partial x =$ constant) and convective boundary conditions for heat transfer $-k(\partial T/\partial x) = h(T_\infty - T_0)$.

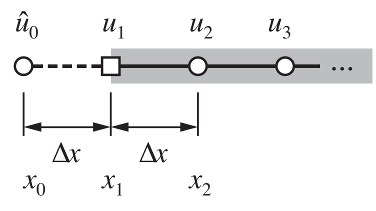
The simplistic implementation is to replace the derivative in Equation (1) with a one-sided difference

$$\frac{u_2^{k+1} - u_1^{k+1}}{\Delta x} = g_0 + h_0 u_1^{k+1} \tag{2}$$

Don't do that! The one-sided difference approximation has a spatial accuracy of $\mathcal{O}(\Delta x)$.

Introduce a Ghost Node

Imagine that there is a node $\hat{u}_0$ that is *outside* of the domain



this node is used to enforce the boundary condition from Equation (1). The value $\hat{u}_0$ does not explicitly appear in the numerical scheme.

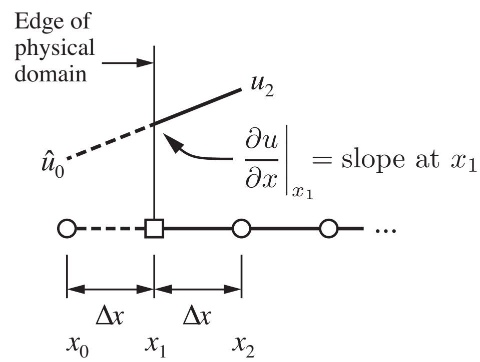
Use the BC to compute $\hat{u}_0$ by extrapolation

Use a *central* difference approximation at $x = 0$ ($x = x_1$) to impose the boundary condition.

$$\frac{u_2 - \hat{u}_0}{2\Delta x} = g_0 + h_0 u_1. \tag{3}$$

The value of $\hat{u}_0$ consistent with the boundary condition is

$$\hat{u}_0 = u_2 - 2\Delta x(g_0 + h_0 u_1). \tag{4}$$



Equation for u_1

Evaluate the finite difference form of the heat equation at $x = x_1$.

$$\frac{u_1^{k+1} - u_1^k}{\Delta t} = \theta\alpha\left[\frac{\hat{u}_0^{k+1} - 2u_1^{k+1} + u_2^{k+1}}{\Delta x^2}\right] + (1-\theta)\alpha\left[\frac{\hat{u}_0^{k} - 2u_1^{k} + u_2^{k}}{\Delta x^2}\right]$$

Choose $\theta = 1/2$ and use the formulas for $\hat{u}_0$ at time step k and time step $k+1$

$$\begin{aligned}\frac{u_1^{k+1} - u_1^k}{\Delta t} = \frac{\theta\alpha}{\Delta x^2}&\left[\boxed{u_2^{k+1} - 2\Delta x(g_0^{k+1} + h_0^{k+1}u_1^{k+1})} - 2u_1^{k+1} + u_2^{k+1}\right]\\ &+\frac{(1-\theta)\alpha}{\Delta x^2}\left[\boxed{u_2^{k} - 2\Delta x(g_0^{k} + h_0^{k}u_1^{k})} - 2u_1^{k} + u_2^{k}\right]\end{aligned}$$

The terms in boxes are from the boundary condition

Rearrange the Equation for u_1

Algebraically rearranging the preceding equation gives

$$a_1 u_1^{k+1} + b_1 u_2^{k+1} = d_1 \tag{5}$$

where

$$a_1 = \frac{1}{\Delta t} + \frac{2\theta\alpha}{\Delta x^2}(1 + \Delta x h_0^{k+1}) \tag{6}$$

$$b_1 = -\frac{2\theta\alpha}{\Delta x^2} \tag{7}$$

$$\begin{aligned} d_1 = &\left[\frac{1}{\Delta t} - \frac{2(1-\theta)\alpha}{\Delta x^2}(1 + \Delta x h_0^k)\right] u_1^k \\ &+ \frac{2(1-\theta)\alpha}{\Delta x^2} u_2^k - \frac{2\alpha}{\Delta x}\left[\theta g_0^{k+1} + (1-\theta) g_0^k\right] \end{aligned} \tag{8}$$

These equations define the terms for the first row in the system of equations

Data structure for implementing alternative BC in the MATLAB code

Store the boundary condition specification in a 2×3 matrix. The first row has data for $x = 0$ and the second row has data for $x = L$.

The first column is a flag with the boundary condition type.

$\texttt{b} = 1$ at the $x = 0$ boundary and $\texttt{b} = 2$ at the $x = L$ boundary

$\texttt{ubc(b,1)} = 1$: $u(x_b) =$ value
$\texttt{ubc(b,2)} =$ value of u at boundary
$\texttt{ubc(b,3)} =$ not used

$\texttt{ubc(b,1)} = 2$: $\partial u/\partial x|_{x_b} = g + hu(x_b)$
$\texttt{ubc(b,2)} = g$
$\texttt{ubc(b,3)} = h$

The code is in `heatCNBC`

Verification: Solve the toy problem on half of the domain

The toy problem used to test the codes

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 \le x \le L$$
$$u(0,t) = u(L,t) = 0;$$
$$u(x,0) = \sin(\pi x/L)$$

only needs to be solved on one half of the domain

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad t > 0,\ 0 \le x \le L/2$$
$$u(0,t) = 0; \quad \left.\frac{\partial u}{\partial x}\right|_{L/2} = 0$$
$$u(x,0) = \sin(\pi x/L)$$

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad t > 0,\ L/2 \le x \le L$$
$$\left.\frac{\partial u}{\partial x}\right|_{L/2} = 0 \quad u(L,t) = 0;$$
$$u(x,0) = \sin(\pi x/L)$$

Verification: Solve the toy problem on half of the domain

Output of `demoCNBC`